VI.—CRITICAL NOTICES.

Collected Logical Works. Vol. II. *Laws of Thought.* GEORGE BOOLE. Open Court Company. Edited by P. E. B. JOURDAIN. Pp. xvi, 448.

THIS work, which has been very rare and consequently but little read, is now being published by the Open Court Company under the editorship of Mr. Jourdain. At present only the second volume has appeared; but this contains Boole's *magnum opus—The Laws of Thought*—and we are promised the first volume with Mr. Jourdain's introduction shortly.

I have no hesitation in saying that this book is one of the most fascinating that I have ever read. It is a delight from beginning to end; its long period of obscurity has been a real misfortune to logic; and the Open Court Company is to be congratulated on making it accessible and putting the editorship into the hands of one whose name is a sufficient guarantee of his eminent capacity.

For a work of this kind the present volume is creditably free from misprints; but I have noticed some, and there are probably others which I have overlooked. On page 151 we read of 'the constituents in the development of y' where y is clearly a misprint for V. In the second formula on page 286 Prob. c in the denominator should be Prob. C. In the last two equations on page 302 a symbol z occurs where, to be consistent with equation (3) on the same page, w should appear. There is a curious error on page 317. Boole is trying to find the major numerical limits of the expression $xy + x(1 - y)z$. He proves that these must be $n(x)$ and $n(y) + n(z)$ and then adds 'of these two values the last, supposing it to be less than $n(1)$, must be taken'. This must be a mistake. We must take whichever is the less of the two expressions $n(x)$ or $n(y) + n(z)$; and the fact that $n(y) + n(z) < n(1)$ does not involve that $n(y) + n(z) < n(x)$.

It is indeed easy to make up an example when this is false. Suppose that $n(1)$ is the number of male human beings, that $n(x)$ is the number of German men, $n(y)$ the number of red-haired men, and $n(z)$ the number of soldiers. Then it is tolerably obvious (*a*) that $n(y) + n(z) < n(1)$, and yet (*b*) that $n(x) < n(y) + n(z)$. Probably the true explanation of the passage is that *last* is a misprint for *least*, which saves Boole's logic at the expense of his grammar.[1] On the top of page 323 we get the equation Min. lim. $D \overline{\gtrless} n(1)$. It seems clear that this is a misprint for Min. lim. $n(D) \overline{\gtrless} 0$, an equation which actually appears on the middle of the previous page.

Whilst I regard Boole's work as a great intellectual achievement, I think it is stronger mathematically than philosophically. Perhaps the most important part of it is the sketch of a general method of dealing with problems in probability. In many respects Boole's system has undoubtedly been surpassed by later logical writings such as those of Frege, Peano, Russell, and Whitehead, etc. My best plan will be to begin with a summary of the *Laws of Thought*, and then to mention some points where I disagree with it and to compare its merits and defects with those of some outstanding modern system of symbolic logic such as *Principia Mathematica*.

Logic, according to Boole, deals with the laws of our mental operations. These are determined by observation, yet our knowledge of these laws differs in kind from the knowledge of the laws of nature which we reach by observation and induction. The latter knowledge is only probable, and its probability continually increases as we become acquainted with more and more numerous favourable instances. But when we observe the operations of our own minds we become aware of a general law in the particular cases, and, once we clearly perceive it, no amount of additional instances will add to the strength of our belief. Boole does not call such knowledge *à priori*, because it does depend in a certain way on experience; but it is undoubtedly *à priori* in the sense of Kant or Meinong and in the only reasonable sense of the word. A knowledge of these laws will enable us to deal (*a*) with relations between things, and (*b*) with relations between facts or propositions. We shall thus be able to give a theoretical solution of the most general problems in ordinary logic and in probability, and we may hope in the end to obtain some light on the constitution of the human mind.

The most general problem of logic is: Given any number of relations between any number of terms x, y, z . . . to deduce all that we can as to the relations between any other set of terms u, v, w . . . which may or may not be wholly or partly identical with the first set. x, y, z . . . u, v, w . . . may here be either simple or complex. The most general problem of probability is: Given the probabilities ot any set of events subject to any set of conditions to determine those of any other set of events subject to any other set of conditions.

That logical operations can be represented by symbols is a fact which may be suspected when we recognise that all language is symbolism. That these symbols will obey laws very similar to those of algebraic symbols is a further fact which may be discovered either by considering the implications of language or by appealing

[1] I owe this conjectural emendation to Prof. Taylor.

directly to our mental operations. Thus, if single letters like and y stand for the class of objects to which the *name* x or the *name* y is applicable; if the combination xy stands for the class of objects to which the *name* x and the *name* y are both applicable; and if the symbol $x + y$ stands for the group of objects to which the *name* x is applicable together with the group to which the *name* y is applicable, a mere consideration of the use of language will tell us that

$$xy = yx \quad \text{where = represents identity of membership.}$$

$$x + y = y + x$$

and

$$z(x + y) = zx + zy$$

which are perfectly comparable to the fundamental laws of algebra. It will also tell us that

$$x \,.\, x \text{ (or } x^2) = x$$

a law which is peculiar to logic and is only true in algebra if x be restricted in value to 0 or 1.

Boole then proceeds to deduce these laws by direct consideration of the operations of the human mind. In his view the hearing or seeing of a general name causes the mind to turn its attention to a certain restricted group in an universe of discourse which is already before it. All the laws can be deduced from considering such operations and their combinations. He holds that in reasoning signs stand *directly* for conceptions and operations of the mind, but that, since these themselves represent things and their relations, signs *indirectly* stand for the latter. And all propositions are properly expressed by equation, in Boole's opinion; for all verbs can be reduced to the identification of two classes.

Boole makes his symbolism as like that of ordinary algebra as possible; and he does this intentionally. He says that the similarity of the formal laws, apart from questions of interpretation, is enough to justify a common symbolism. Really in short his plan is to treat logical formulæ exactly as if they were algebraical ones and to perform all intermediate processes as if this were true. At many intermediate stages this leads to logically uninterpretable equations, but at the end by subjecting the result to the condition $x^2 = x$, which differentiates logic from algebra, interpretable formulæ are obtained. Boole also takes over the numerical symbols 0 and 1, and shows that, if they are to have analogies in logic to their characteristic properties in algebra, *viz.*,

$$0.x = 0$$

$$1.x = x,$$

0 must stand for the null-class, and 1 for the universe of discourse.

He seems to hold (*a*) that the justifiability of using uninterpretable processes which lead to interpretable and true results is guaranteed *à priori* (in our sense, though not in his). That is when one has observed its success in a certain number of cases one sees that it is always justifiable, and sees this with complete certainty.

(*b*) He also holds that unless this procedure were justifiable there would be little use in attempting to deal mathematically with logic. (Cap. V. §§ 2-6.)

Boole's general method in logic may be summed up as follows: (*a*) If you are given an arbitrary function of classes x, y, . . . it may not be logically interpretable. But any statement expressed as an equation will, after certain transformations, be logically interpretable. (*b*) Let $V = 0$ be any equation, the left-hand side of which is of the form $\phi(x, y, z \ldots)$ when $x, y, z \ldots$ stands for classes and ϕ is any mathematical function. If we treat this simply as an algebraic expression in which the variables are restricted in their values to 0 or 1 we can always expand it in the form

$$\phi(1, 1, 1 \ldots)xyz \ldots + \phi(0, 1, 1 \ldots)(1 - x)yz \ldots + \ldots$$

where the variable factors (which Boole calls *constituents*) consist of all the combinations that can be formed by picking out 0 or 1 or . . . n of the n variables, forming their product, and multiplying it by the products of unity diminished in turn by each of the variables that has been left out in the first part of the process. The expansion will thus contain 2^n constituents, and it is obvious (1) that the product of any two of these vanishes, since any product of the form $x(1 - x)$ equals 0; (2) that the sum of all of them $= 1$; and (3) that they really represent a complete dichotomous division of the whole universe of discourse with respect to the properties for which the variables stand. Finally any constituent whose coefficient in this expansion is not 0 must be equated to 0, and each of these equations is a logically interpretable proposition. (This follows from the two facts (1) that the product of two different constituents $= 0$ and (2) that the square of any constituent is equal to the constituent.) (*c*) Again any explicit equation of the form $u = \phi(x, y, z \ldots)$ is logically interpretable, even though $\phi(x, y, z \ldots)$ itself be not so. When the right-hand side is expanded it will appear as a series of constituents whose coefficients are either 1, 0, $\frac{0}{0}$, or a, where a is any coefficient other than these and including as a limiting case $\frac{1}{0}$. (1) The interpretation of 1 and 0 (the two coefficients which obey the law $a(1 - a) = 0$) presents no difficulty. (2) If $a(1 - a) \neq 0$ it can be proved that the constituent whose coefficient is a must be equated to 0. (This is simply an application of the fact that u, since it represents a class, is subject to the condition that $u^2 = u$.) (3) The coefficient $\frac{0}{0}$ cannot be interpreted by means of mathematical deductions; it can be seen, however, that whenever $\frac{0}{0}$ appears as the coefficient of any constituent in the expansion of u the interpretation is that u contains an undetermined proportion of that constituent. To take a simple example:—

$u(1 - x) = 0$ obviously expresses the fact that All u is x.

Solving for u we get $u = \dfrac{0}{1 - x}$

$$= \frac{0}{0}x + \frac{0}{1}(1 - x)$$

$= \dfrac{0}{0}x$, when the interpretation clearly is that u is identical with an indefinite part of x.

There is one further point of interest to notice here. We see that the constituents in the expansion of u whose coefficients do not obey the equation $a(1 - a)$ must be separately equated to 0. What does this mean? It means that *if* u, which was perhaps given implicitly in an expression of the form $\phi(x, y, z \ldots u) = 0$, is to be capable of representing a class at all, certain relations must hold between $x, y, z \ldots$ etc. These relations were not explicit before, but they become so when the equation is solved for u and the conditions which distinguish logic from mere algebra are imposed on the solution.

(*d*) Boole is now in a position to tackle his general logical problem. For this purpose two further processes are needed: (1) what he calls *Reduction, i.e.*, the combination of the premises into a single proposition, and (2) *Elimination, i.e.*, the removal of terms which are present in the premises but are not needed in the conclusion. It is proved that if our premises be put in the form $V_1 = 0$, $V_2 = 0 \ldots V_n = 0$, then the equation $V_1 + c_2V_2 + \ldots c_nV_n = 0$ (when $c_2 \ldots c_n$ are arbitrary multipliers) gives all the information provided by the separate premises and no more. Again if the coefficients of the constituents in the expansions of $V_1, V_2. V_n$ be all positive, the coefficients $c_1, c_2 \ldots c_n$ can all be reduced to unity, and $V_1 + V_2 + \ldots V_n = 0$ will give all and no more than all the required information. Lastly, if these coefficients be not all positive it is only necessary to square each of the equations and add. So that $V_1^2 + V_2^2 + \ldots V_n^2 = 0$ will always have just the combined force of the premises $V_1 = 0$, $V_2 = 0, \ldots V_n = 0$. (These results are once more a consequence of the fundamental facts that if t_m and t_n be any two constituents $t_mt_n = 0$ and $t_m^2 = t_m$ and $t_n^2 = t_n$.)

Elimination, as Boole carefully points out, is considerably different in logic and in algebra. In algebra the number of terms that can be eliminated depends on the number of independent equations between them that are given. But in logic elimination is conducted by means of the fundamental equation of duality $x^2 = x$, and so any number of terms can be eliminated even from a single equation. (The only limitation is that, if you try to eliminate so many terms that your original data supply no information as to the relations between those that are left, you will be confronted with the platitude $0 = 0$.) The result of eliminating x from

$\phi(x) = 0$ is $\phi(1) . \phi(0) = 0$. That of eliminating x from $\phi(x, y) = 0$ is $\phi(1, y) . \phi(0, y) = 0$. That of eliminating x and y from $\phi(x, y, z) = 0$ is $\phi(1, 1, z) . \phi(1, 0, z) . \phi(0, 1, z) . \phi(0, 0, z) = 0$.

The general rule can easily be seen from these examples. The proof depends on expansion in constituents and application of the Law of Duality $x(1 - x) = 0$.

(*e*) The solution of Boole's general logical problem is now all over except the shouting (which in this case consists of certain methods for abbreviating the process described above). The problem is: Given premises involving classes $x, y \ldots$ to find all that can be discovered from them about any class u which is any function of the classes $z, w \ldots$. (It is not necessary that $z, w \ldots$ etc., should explicitly be included among the $x, y \ldots$ of the premises, for they can always be introduced on expansion in constituents, *e.g.*, $x = xw + x(1 - w)$.) The solution of the problem is: (1) Reduce the premises to a single equation $\phi(x, y \ldots) = 0$. (2) If $\psi(z, w \ldots)$ be the required function put $u - \psi(z, w \ldots) = 0$. (3) Reduce these two equations to a single one of the form $\chi(x, y \ldots; z, w \ldots; u) = 0$. (4) Eliminate $z, w \ldots$ from this. (5) Solve the resulting equation for u. You will thus obtain u as an explicit function of constituents involving $x, y \ldots$.

This will be an interpretable proposition, and any necessary conditions among the variables $x, y \ldots$ will become explicit. In Chapter IX. Boole gives various methods by which these processes may be shortened. They consist essentially in recognizing the simplifications which the Law of Duality imposes on algebraic expressions. Thus our old friends

$$pVp . \equiv . p \text{ and}$$
$$(pVq)\,(pVr) . \equiv . pVqr \text{ and}$$
$$pVpq . \equiv . p$$

appear here under thin disguises. What this chapter really tells us is that it is often useful even for practical purposes *not* to wait till the end of a process before imposing the conditions that differentiate logic from algebra. As we shall see later, this is rather an important admission.

The next important point to notice is Boole's distinction between primary and secondary propositions and his method of dealing with the latter. A primary proposition for Boole is one which makes an assertion about things, *e.g.*, *Cæsar crossed the Rubicon*, *All men are mortal*, etc. A secondary proposition states a relation between facts, *e.g.*, *If it rains I shall get wet*, *Either he will arrive by* 2.30 *or I shall go out*. Not all propositions in the hypothetical or disjunctive form are secondaries. Boole calls: *Animals are either rational or irrational* primary. And not all secondaries, according to him, are hypothetical or disjunctive. *It is true that Smith is a knave* would be a secondary proposition. Boole treats all secondary propositions as referring to time. Let X, Y . . . be primary propositions. Let x be the class of moments at which x

is true; similarly for y, etc. Let 1 stand for the whole time under consideration. Then (*a*) X *is true* can be expressed by $x = 1$, and x *is false* by $x = 0$. *If* Y *then* X can be expressed by $y = vx$ when v is an indeterminate class of moments which may have any value from 0 to 1. *Either* Y *is true or* X *is true* can be expressed by $y + (1 - y)x = 1$. (*b*) Equations containing x, y, obey all the laws of primary propositions, and in our work we can forget their reference to time and act as if we were dealing solely with primary propositions.

Boole's book teems with examples fully worked out, which are of great use to the student. He devotes two chapters (XIII. and XIV.) to a full treatment of certain arguments used by Clarke and Spinoza. There is also an interesting chapter (XV.) on Aristotelian Logic. Boole easily deduces the rules for immediate inferences and for the syllogism on his principles. The syllogism is essentially a method of elimination. Boole argues that probably all elimination could be reduced to syllogism, but that the general problem of logic is not merely one of elimination but is the one which he has solved; and that the solution of this general problem cannot be performed by the doctrine of the syllogism alone. Moreover, he says, whilst such principles as the *Dictum de Omni et Nullo* are self-evident, they are not fundamental. They, together with much else which cannot be deduced from them, can be deduced from more primitive principles. He is inclined to make the Law of Contradiction the fundamental principle of logic. This is apparently because we have constantly used the Law of Duality, $x^2 = x$, and because this can be written in the form $x \,.\, x = x \,.\, 1$, whence $x \,.\, x - x \,.\, 1 = 0$ or $x(1 - x) = 0$. And this, on interpretation, becomes: Nothing is both x and not x.

It will be convenient to criticise Boole's logical doctrines before passing to his theory of probability. The latter is based on the former, but involves additional elements which will need to be explained and criticised later.

(1) Is logic really the science of the laws of our mental operations? Boole continually speaks as if it were. Yet he certainly does not confuse it with empirical psychology, since he holds that the truth of its laws is seen in their instances, not merely rendered probable by induction. And, as far as I can see, the only positive argument that he produces for thinking the laws of logic to be laws of our mental operations and not laws of their objects is that such an axiom as

$$xy = yx$$

involves a difference of order which is present among our acts of thought, but is not present among their objects. To me it seems clear that this argument does not show that the laws of logic are laws of our mental operations, and that the truth is that they are the laws of certain objects, *viz.*, propositions, their parts, and their relations.

It is true that these objects are essentially objects of *thoughts* (as distinct from other mental states such as sensation and perception), and further that the relation which subsists between the objects of certain acts of thought (*e.g.*, in the case of inference) determine whether these acts can be described as *valid* or *justifiable*. But this seems to be the sole special connexion between logic and thought, and it evidently does not make the laws of logic laws of thinking. As to the equation $xy = yx$ the truth seems to be as follows: As a matter of physical fact two symbols x and y can be written in two different orders; as a matter of psychical fact two classes can be thought of successively in two different orders; as a matter of logical fact the symbols xy and yx stand for one and the same class. The law $xy = yx$ is therefore a statement that there is no difference among *logical* objects to correspond to the difference of *spatial* order among symbols, or to the difference of *temporal* order among acts of thinking. So far then from being purely a law of thought, as Boole suggests, the identity asserted by it can only be understood if we go outside the different and successive acts of thinking and consider their single and timeless logical object. (Similar considerations would show that it cannot be an assertion purely about symbols.)

(2) Should propositions be expressed by equations? In the main undoubtedly Boole's motive for expressing propositions as equations was to obtain as much analogy as possible with ordinary algebra. The same may be said of his treatment of secondary propositions. A logician who is breaking new ground in formal logic will always be torn between two ideals: (*a*) that of recognising every distinction among propositions and of analysing the different kinds as fully and accurately as possible, and (*b*) that of establishing a symbolism which shall be as simple and fruitful as possible. Frege and Boole illustrate the striving after the first and the second of these ideals respectively, and it is the merit of Russell's and Whitehead's system to hold the balance very evenly between them. In general Boole does not pretend that an equational representation is an adequate analysis of all kinds of propositions, yet he does seem to offer one argument. In trying to show that all verbs may be replaced by = and a class-symbol he argues as follows: You cannot understand the proposition *Cæsar conquered the Gauls* unless you understand what is meant by *One who conquered the Gauls*. Hence the latter is an essential constituent of the former, and the proposition really *means* (and is not merely logically equivalent to) *Cæsar is identical with one who conquered the Gauls*. The error of this analysis seems to me to be that it overlooks the important fact that a *finite* verb has two logical functions. As a *verb* it represents a relation, the same relation as its infinitive stands for. As a *finite* verb it also makes an assertion, the sign of which is the verbal inflexion. Now the fact of assertion is indeed common to all propositions whatever, and could be re-

presented by a common word or symbol the same for all propositions. But the relations represented by different verbs are different from each other and different from that represented by the verb *to be* (as used in *Cæsar is mortal*). If you force the verb into the predicate as in *One who conquered the Gauls* you have made no analysis whatever. You have (*a*) to recognise that this is at least a very different kind of predicate from *mortal* or even from *one who* is *mortal;* and (*b*) that it involves a relation between terms which is not that of identity. The verb has merely gone into the grammatical predicate, and any attempt to get rid of it there in favour of the relation of identity will only start you on an infinite regress. To put it generally, the notion of x's being mortal seems to be logically prior to the notion of *an x such that x is mortal* and it is therefore perverse to offer *Smith = one who is mortal* as an analysis of *Smith is mortal;* and further, even if this analysis were valid, it is mere lack of thought to treat *Cæsar conquered the Gauls* as if it were probably similar to *Cæsar is mortal.*

(3) There is nothing then to be said for equations as an *analysis* of propositions in general. Can we say that the advantages of making formal logic as analogous as possible to algebra outweigh the disadvantages? The only advantage that I can see is that elementary algebra and its symbolism is familiar to all educated people. Against this we may set the following disadvantages: (*a*) As we have seen an equational system necessarily involves a divorce between formal development and philosophical analysis. (*b*) Experience shows only too clearly how liable the practice of using the same symbols to represent different kinds of objects is to lead to hopeless confusion, *viz.*, to the failure to recognise that the objects denoted by the same symbols are different. The sign *2* unfortunately represents the integer 2, the rational fraction $\frac{2}{1}$, and the real number 2 (*i.e.*, a class of rational fractions). These are utterly different things; but, owing to the fact that they are all represented by the same sign, it is extremely difficult to get most people to see that they differ. (*c*) If formal logic be used (as in *Principia Mathematica*) for proving the fundamental laws of arithmetic, and, more generally, if we want to determine the relation of logic to mathematics, our enquiry will be confused and prejudiced at the outset by using in logic the symbols of arithmetic. (*d*) Since the mathematics of logic is simpler than ordinary algebra (owing to the existence of the relation $x^2 = x$ in logic) it is very perverse to insist on pretending all through one's work that this simplification is absent and only to impose it at the end. Boole himself practically admits this when he introduces his chapter on *Methods of Abbreviation.* (*e*) If we work with implications instead of equivalences we can always get back to equivalences if we want them by using the two equations

$$p)q \,.\equiv:\, p \,.\equiv.\, pq$$

and

$$p)q :\equiv:\, q \,.\equiv.\, p\vee q.$$

(4) A defect in Boole's logical symbolism is his treatment of particular propositions. We very greatly miss the symbols $(\exists x)$ and (x) of Russell and Whitehead. Primarily letters in Boole's system represent classes; thus we may compare his x's and y's to Russell's and Whitehead's α's and β's. But, owing to his having no symbol for class-membership, no symbols for individuals, no incomplete symbol like $(\exists x)$ . . ., and consequently no expressions of the form $(\exists x) . x \epsilon \alpha$, he is faced with the following problem: He must express *particular* propositions solely by relations of *equality* among *classes*. To do this satisfactorily is almost impossible. Schröder,[1] whose system resembles Boole's in many respects, used inequalities. But Boole does not do this in his logic, presumably from his desire to keep as close to ordinary algebraic equations as possible. He thus expresses All y is x by the equation $y = vx$, which means: The class y is identical with the common part of the class x and some indeterminate class v. What Boole really wants to say is that All y is x is equivalent to the statement: There is a class v such that $y = vx$. But he has no means of symbolising this kind of statement. In Russell's and Whitehead's notation it would be expressed in the form $(\exists v) . y = v \frown x$; and this is formally equivalent to $y \subset x$. Having no symbol such as $(\exists v)$ Boole is compelled to introduce his indefinite class-symbol v as a real variable instead of an apparent variable. There is nothing in the nature of his symbolism to show that v, rather than x or y, stands for: *There is a class v such that* . . ., and that the statement $y = vx$ is not about the class v in the same sense in which it is about the classes x and y.

This defect is not very important in dealing with A propositions, because, as Boole points out, v can be eliminated and *All y is x* can be expressed by the equation $y(1 - x) = 0$. But this excuse cannot be made for his symbolism for I and O propositions. He symbolises Some y is x by the equation $vy = vx$. Allowing that v may be interpreted as *There is a class v such that* . . ., this means: There is a class such that its common part with x is identical with its common part with y. But this will always be true; for, if v be the null-class, we have $y.0 = x.0$ whatever x and y may be. We ought therefore at least to add the statement that $v \neq 0$. Hence a particular proposition cannot be expressed by an *equality* among classes alone. Again, we might enquire why v should appear on both sides. Would not the equation $vy = wx$, *i.e.*, There is a class v whose common part with y is identical with the common part of x and some class w, be more general? Take: Some men are black. If this be true is it certain that there is any *one* class except the null-class and the class of black men such that its common parts with x and with y are identical? Neglecting

[1] Schröder and Couturat have also incomplete symbols Π and Σ to correspond to (x) . . . and $(\exists x)$. . . .

the first case, for the reasons given above, the second would reduce the equation to the tautology $xy.y = xy.x$

i.e., $xy^2 = x^2y$

or $xy = xy$ by the Law of Duality.

To illustrate the same point and to give an example of Boole's methods we may solve the equation $vy = vx$ for y. We have

$$y = \frac{v}{v}x.$$

Whence $$y = vx + (1 - v)x \cdot \frac{0}{0} + (1 - v)(1 - x) \cdot \frac{0}{0}.$$

Here, it will be observed, a new indeterminate class symbol has been introduced by the coefficient $\frac{0}{0}$. And it cannot in general be identical with v itself, or the equation would reduce to $y = vx$ which represents All y is x.

As Boole points out v cannot be eliminated from $vy = vx$, for the attempt leads to the platitude $0 = 0$. As he also points out either v or w but not both can be eliminated from $vy = wx$, the result being, *e.g.*, $vy(1 - x) = 0$. This form of the equation is also open to the criticism mentioned above that, unless the *inequality* $v \neq 0$ be added, it does not properly represent a particular proposition. In fact a comparison of this form of the equation with the equation $y(1 - x) = 0$, which represents an A proposition, shows very clearly that Boole is trying to represent an I proposition by an A proposition; for $vy(1 - x) = 0$ means literally All vy is x, and, allowing for Boole's failure to symbolise *There is a v such that* . . ., means: There is a v such that all vy is x. This must be admitted to be a very clumsy and unnatural way of symbolising *Some y is x.*

There is one other point to notice before leaving this subject. When, in the solution of an equation, Boole gets several constituents each provided with the coefficient $\frac{0}{0}$, he tells us that we may add the constituents as they stand and prefix $\frac{0}{0}$ to the result. The reason given is that $\frac{0}{0}$ may stand for *any* class. This is surely a bad reason. If it may stand for any class we are not justified in assuming that it stands for the same class in each case; and unless this be assumed it is not obvious why we may take it outside the sum as a single logical factor. Boole's procedure is, however, really justifiable. Suppose we have such an equation as

$$y = \frac{0}{0}x + \frac{0}{0}(1 - x)z$$

or, as it might be written, $y = vx + w(1 - x)z$. The question is whether we are justified in writing this in the form

$$y = w(x + \overline{1 - x}z) \text{ or } y = \frac{0}{0}(x + \overline{1 - x}z).$$

Boole could have proved the justifiability of this procedure in the following way: If we put $y = vx + w(1 - x)z$ and $t = x + (1 - x)z$ we can form a single equation. If we eliminate from this v, w, x, and z we shall find ourselves left with the equation $y(1 - t) = 0$, which, on solution, gives $y = \frac{0}{0}t$, *i.e.*, $y = \frac{0}{0}(x + \overline{1 - x}z)$. The very fact, however, that there is an apparent difficulty here shows clearly that symbols like u, v, $\frac{0}{0}$ are not ordinary class-symbols like x, y, z . . ., but are a very awkward and inadequate way of symbolising what Russell and Whitehead denote by the incomplete symbol $(\exists u)$. . . . Thus the proposition $y = vx + w(1 - x)z$ is really only adequately symbolised by the expression

$$(\exists v, w) \,.\, y = vx + w(1 - x)z.$$

(5) The last point that I shall criticise before leaving the purely logical part of the work is the distinction between primary and secondary propositions and the introduction of time. In the latter point Boole is once more followed by Schröder, and it seems to me that, apart from all special arguments, a comparison with the respective fates of Fluxions and the Differential Calculus is ominous for this procedure. There is undoubtedly a genuine distinction between primary and secondary propositions, and Boole's distinction partly coincides with it. A proposition which asserts a quality of a proposition or propositional function, or a proposition which asserts a relation between two propositions or propositional functions, may fairly be called secondary. Thus *p is true*, *p is necessary*, $p)q$, $\phi x)_x \psi x$, and $(x) \,.\, \phi x \,.\,) \,.\, (x) \,.\, \psi x$ are secondary propositions. Now Boole so far agrees with this as to call secondary (a) propositions which ascribe the quality of truth or falsehood to propositions, and (b) those which assert a relation of disjunction or implication between two *propositions* (*e.g.*, what Keynes calls 'True Hypotheticals'). Thus he would count as secondary: *If it rains I shall get wet* and *If everybody be unvaccinated somebody will have small-pox*. But (c) he does not count as secondary propositions of the form $\phi x)_x \psi x$, *i.e.*, what Keynes calls 'Conditionals,' nor the corresponding disjunctives. Thus he would count as primary the proposition *If anyone be unvaccinated he will have small-pox*. There seems to be no good ground for this distinction, and Boole's error doubtless arises from the fact that he did not clearly recognise the distinction between propositions and propositional functions, and between real and apparent variables.

If he had carried his analysis further and declined to regard equations expressing identity between classes as ultimate, he would have seen that primary propositions are really by no means common, and that the greater number of his so-called primary propositions are really assertions about the formal equivalence of functions.

We may now turn to Boole's doctrine of the connexion of secondary propositions with time. Boole seems to regard propositions asserting relations between events as the type of secondary propositions. Now these do contain an essential relation to time. But when he tries to make propositions like *p is true* refer to time his doctrine loses its plausibility. It loses it still further when we remember that a vast number of hypothetical propositions are not about events at all but about essentially timeless objects. Take the proposition if $3 > 2$ and $2 > 1$ then $3 > 1$. It is surely preposterous to offer as the meaning of this: The class of moments at which it is true that $3 > 2$ and that $2 > 1$ is identical with some part of the class of moments at which $3 > 1$. The absurdity is due to the fact that objects like 1, 2, and 3 are timeless, and the relations between them are timeless too.

Boole explicitly identifies eternal truths with propositions which are true at all times. This appears to me to contain a double error. (*a*) All propositions, if true at all, are true independently of time. When we say that a proposition about x is sometimes true we mean that a *function* involving x and t gives true propositions for certain values of t. This is disguised by the facts (1) that all assertions about events really involve a reference to the time at which they happen, and (2) that this reference is often not made explicit in speech and writing. Thus *Queen Anne is dead* seems to stand for a *proposition* and to be true at some times and false at others. But the fact is that, since the death of Queen Anne is an event, this form of words is incomplete, for it contains no explicit reference to time. The same form of words as used by me and as used by William III. do not stand for the same proposition, and therefore the fact that my statement would be true and William's verbally identical statement false does not prove that any proposition has been false and has become true. (*b*) A proposition which is 'always true' is an assertion that a function involving time gives true propositions for all values of t. Thus the proposition *If amber be rubbed with silk it becomes electrically charged* means *If at any time amber be rubbed with silk it then becomes charged at that time.* Such propositions are always about events. An eternally true proposition is one about the timeless qualities or relations of timeless objects. The whole of pure mathematics and logic provides an example of this.

Boole's own treatment of the relations of propositions to time seems to me very unsatisfactory and confused. He writes for X *is true* $x = 1$, *i.e.*, The times at which X is true are all times. But he also holds that a proposition may be sometimes true and sometimes false. How can this be compatible with the above notation for X *is true?* I suppose the solution is that for Boole X *is true* has two senses. (*a*) It is an incomplete symbol which only stands for a proposition when a temporal determination is added. (*b*) It is this with the determination *at all times* added. He nowhere gives an expression for X *is sometimes true.* I suppose it would have to be $x = v$ and $v \neq 0$.

Let us now pass to Boole's general method in probability. As before we will first state and then criticise. According to him probability may be approached from two different points of view; each will lead to the same numerical results, and each in the end needs to be supplemented by the other. The first method is to define probability fractions as the ratio of the number of cases that give true values to a given propositional function (Boole does not of course use this expression) to the total number of cases, assuming them all to be equally likely. With this definition we can prove the usual fundamental propositions about the probabilities of conjunctive and disjunctive propositions, and we shall find that the probability of any event compounded in any way of the simple events x, y . . . is the same *algebraical* function of their separate probabilities p, q . . . as the compound event is a *logical* function of the events x, y. . . . The other method of attack is to start by assuming that expectation is a state of mind which, although it cannot be accurately measured, is at least subject to certain rules of increase and decrease. If we now assume that the measure of the expectation of a complex event is the same *algebraical* function of the probabilities of the separate events as the expression for the complex event is a *logical* function of the separate events, we find (*a*) that what common-sense judges to be greater or less degrees of expectation will have greater or less measures respectively, (*b*) that certainty is expressed by 1, and (*c*) that the ordinary laws of probability follow.

We now come to Boole's solution of the general problem. By 'the event x' he means 'that event of which the proposition which asserts the occurrence is expressed by the equation $x = 1$'. And similarly for compound events. Events are 'conditioned' when they are not free to occur in every possible combination; otherwise they are unconditioned. If now $\phi(x, y, z) = 1$ represents a compound event; x, y, z represent simple *unconditioned* events; and the probabilities of x, y, z, etc., are p, q, r . . . respectively, then Prob. $\phi(x, y, z) = \phi(p, q, r \ldots)$ when the $+$'s, $\times$'s, etc., in the first are to be read in their logical sense, and in the second in their arithmetical sense. Next Boole determines the *unconditioned* probabilities of a number of simple events given their probabilities under a condition $V = 1$. Now let x, y, z be any simple events; let S, T . . . be any compound events which are logical functions of these, and let us try to find the probability of any other compound event W. We can form a logical equation expressing W in terms of constituents formed from S, T, etc., regarding these as single logical terms. It will take the form

$$w = A + 0B + \frac{0}{0}C + \frac{1}{0}D$$

when A, B, C, D are sums of constituents involving s, t, . . . etc. (Here w, s, t . . . are letters written instead of the complex functions W, S, T, etc. What we do is to write s = S, t = T . . ., w = W; reduce these to a single logical equation, and eliminate x, v, . . .) The solution of the above logical equation is

$$w = A + qC$$

and

$$D = 0.$$

The latter is a condition independent of w and may also be written in the form A + B + C = 1, or, for shortness, V = 1.

We now wish to pass from logic to probability. We were given the probabilities of s, t, . . .; but the condition V = 1 has emerged as involved in our data. Hence the given probabilities were probabilities *subject to the condition* V = 1 and not the probabilities of s, t, . . . as *unconditioned* events. We cannot therefore pass at once from logic to algebra, but must first find the *unconditioned* probabilities p^1, q^1, . . . of these events by the method which Boole has already given us. If we substituted these values straight away on the right-hand side of our equation, we should get the probability of w as an *unconditioned* event; but w is not unconditioned for it is subject to the condition V = 1. Hence we really require to find Prob. w under the condition V = 1. This Boole shows to be equal to $\frac{\text{Prob. V}w}{\text{Prob. V}}$. Hence:

$$\text{Prob. } w = \frac{\text{Prob. V(A} + q\text{C)}}{\text{Prob. V}} = \frac{\text{Prob. (A} + q\text{C)}^1}{\text{Prob. V}}.$$

The right-hand side can now be determined by substituting the values p^1, q^1, . . . for s, t, . . . respectively everywhere in it, and reading all *logical* +'s, ×'s, etc., as *algebraical* ones. (I should say that Boole's exposition here is very condensed, and, to me, hard to follow. I think I have understood it, but I have added several steps that seem to me (*a*) justifiable, and (*b*) necessary for clearness.

Boole solves the still more general problem when the probabilities of S, T, etc., are given not *simpliciter* but under an explicit condition. No additional difficulty in theory is involved here since the explicit condition can be dealt with just like the originally implicit one which became explicit in the solution of the simpler problem.

One further question remains if this general method is to lead to determinate results in all cases. In passing from conditioned to the corresponding unconditioned probabilities we may have to solve algebraic equations of a degree higher than the first. We may then be in doubt as to which root to take. In a very

[1] For A, B, C, and D can contain no constituents in common and products of different constituents will vanish.

brilliant chapter on *Statistical Conditions*, Boole shows us how to determine this question. Incidentally in this chapter he gives all that is required for solving problems on Numerically Definite Syllogisms, as De Morgan called them. Examples are then supplied of the general method in two excellent chapters (XX. and XXI.) dealing with Problems on Causes and the Probability of Judgments.

I regard this work of Boole's on probability as being of the utmost brilliance and importance. I am not aware that the general problem which he solves has been solved before or since. So far as I can judge Boole's solution is essentially sound, and perhaps the very neat relation which appears in it between logic and algebra is a good excuse for approximating the two symbolisms, at any rate when dealing with problems on probability. On certain points, however, I find a good deal to criticise.

(1) Boole constantly talks of *the* probability of a proposition. I am sure that this is meaningless or elliptical. Probability is always probability relative to some datum or other. Perhaps *the* probability of a proposition might be interpreted as its probability relative to the laws of formal logic and to no additional propositions; this seems to be what Boole means by unconditioned probability.

(2) Boole confuses two apparently similar but really very different notions, *viz.*, *The probability of p given q* (which, following Mr. W. E. Johnson's convenient notation, we will write p/q) with *The probability of if q then p*. Interpreting *the* probability of any proposition as its probability relative to the laws of formal logic, and denoting the latter by f, this would be written $[q)p]/f$. Now the two are quite different. One is the probability that p is true given that q is true; the other is the probability that q implies p given the laws of formal logic. The cause of the confusion is the following: If we forget that *the* probability of a proposition in itself is meaningless we are liable to think that *The* probability of (p if q) is the same as (The probability of p) if q. And this is what Boole does. It leads him to one very extraordinary conclusion which he himself recognises to be paradoxical and which I regard as in itself a sufficient refutation of his theory. He shows that, on his theory, two formally equivalent propositions will have two different probabilities. The example that he takes is *If x then y* and *Either y is true or both x and y are false*. If the probability of the second be p he proves that that of the first will be $\frac{cp}{1 - p + cp}$ when c is an undetermined constant. Now this result follows through his taking Prob. (if x then y) as $\frac{\text{Prob. } xy}{\text{Prob. } x}$, *i.e.*, taking $[x)y]/f$ as the same as $\frac{xy/f}{x/f}$. But the fact is that they are not equal. The latter $= y/xf$, *i.e*, the probability of y given x and

the laws of logic. The former may be determined by the following considerations :—

$$x)y\,.\equiv.\,\text{not }(x\text{ and not }-y).$$

$\therefore$, assuming that formally equivalent propositions have the same probabilities relative to the same data,

$$\begin{aligned}[x)y]/f &= \text{not }(x\text{ and not }-y)/f\\ &= 1-x\bar{y}/f\ (\text{writing }\bar{y}\text{ for not }-y)\\ &= 1-x/f\,.\,\bar{y}/xf\\ &= 1-x/f(1-y/xf)\\ &= 1-x/f+x/f\,.\,y/xf\\ &= 1-x/f+xy/f.\end{aligned}$$

If we use this value and apply Boole's methods we shall find that $[x)y]/h$, when h is the proposition that $[y+(1-x)(1-y) = 1]/f = p$, is equal to p.

(3) I now pass to a point of intrepretation where I find Boole very difficult to follow. When we solve our general logical equation we get

$$w = A + qC$$

where q is an indeterminate class. When we pass from logic to algebra Boole writes an indeterminate probability c for q. So far all is clear. Then he proceeds to interpret c. I quote his argument (p. 283): 'The logical equation, interpreted in the reverse order, implies that if either A take place or C in connexion with q, w will take place, and not otherwise'. (This is obviously true.) 'Hence q represents the condition under which, if C take place, w will take place. But the probability of q is c. Hence, therefore, c = probability that if C take place w will take place.

Now I cannot accept the latter part of this argument. We have proved (a) that $q\mathrm{C})w$, *i.e.*, that $q\,.\,)\,.\,\mathrm{C})w$. And (b) we are told that $q/f = c$. But the probability of an implied proposition is *not* the same as that of one which implies it on the same data. Suppose, *e.g.*, that $x)y$; let us call this datum h. Let $x/h = p$, and let us try to find y/h. We have

$$\begin{aligned} y &= yx + y\bar{x}.\\ \text{Hence}\qquad y/h &= yx/h + y\bar{x}/h\\ &= x/h\,.\,y/xh + (1-x/h)y/\bar{x}h\\ &= x/h + (1-x/h)y/\bar{x}h\\ &= p + (1-p)q\ \text{when}\ q = y/\bar{x}h.\end{aligned}$$

Hence it does not follow from the facts that $q\,.\,)\,.\,\mathrm{C})w$ and that $q/f = c$ that $[\mathrm{C})w]/f = c$. If, instead of $q\,.\,)\,.\,\mathrm{C})w$, we had $q\,.\equiv.\,\mathrm{C})w$ the required result would be obtained. But we do not have this. If we did we should have to have $\mathrm{C})w\,.\,)\,.\,q$. Now this would imply $w)q$, which is certainly not in general true.

And if we look further into Boole's statements on page 283 we cannot feel sure that he really means to assert that $c = [\mathrm{C})w]/f$.

For he proceeds to add that $c = \frac{Cw/f}{C/f}$. These two statements, as we have seen, are not equivalent; though Boole thought they were. Hence we cannot be sure which of the two he means. I am pretty clear, however, that he means the second. In the first place in the simple example (1) worked out by Boole on page 293, we can see that $c = \frac{Cw/f}{C/f}$. Secondly, I offer with some diffidence the following general proof that $c = \frac{Cw/f}{C/f}$. In the equation $w = A + qC$ multiply both sides by C, remembering that $CA = 0$, and $C^2 = C$. We get $Cw = qC$. Hence $Cw/f = qC/f$. Now, if q and C be independent, $qC/f = q/f \,.\, C/f$. But q is a purely arbitrary proposition; hence its probability cannot be affected by the truth or falsity of C; hence we may treat q and C as independent. We thus get the equation

$$Cw/f = q/f \,.\, C/f$$

i.e.

$$c = q/f = \frac{Cw/f}{C/f}.$$

(4) I find Boole's notation for simple events and conjunctions of simple events far from satisfactory. If x represents the event of raining the proposition *It rains* will be represented by $x = 1$. Similarly if y be the event of thundering, the proposition *It thunders* is represented by $y = 1$; the event xy is the double event of thunder and rain; and the proposition *It thunders and rains* is expressed by $xy = 1$. But if Boole is keeping to his notation for secondary propositions these equations surely ought to stand for the propositions: *It is always raining*, *It is always thundering*, and *It is always raining and thundering* respectively. The fact is that he does not provide a satisfactory notation for the two very different propositions: *It is true that it rains* and *It is always raining*. His failure to provide any notation at all for singular propositions (which, I am afraid, comes from a failure to distinguish the two relations ϵ and $\supset$) is also very inconvenient in dealing with many problems of probability. Nevertheless, I believe that Boole's mathematical treatment of probability is a great and original achievement, and that it would be easy and thoroughly worth while (when we have finished saving civilisation by the mutual slaughter of almost everyone who makes the continuance of civilisation possible) to remove its errors of detail.[1]

I conclude with a few words on Boole's views as to the light that mathematical logic throws on the constitution of the human

[1] I have now (Nov. 1916) succeeded in doing this and in giving a satisfactory account of $\frac{0}{0}$ and C. The work contains too many symbols for its publication in a periodical, so the reader must take my statement on trust for the present.

mind. His most characteristic doctrine is that, whilst the fact that the laws of thought and the laws of matter are mathematical in form might induce us to suppose that mind as well as nature is governed by necessity, the further fact of error shows that this conclusion is unwarranted and that either the mind can break the laws of thought or at least that these laws are only part of a much larger system of laws and may be suspended in the same kind of way in which you may say that the law of gravitation is suspended by the Principle of Archimedes in the case of a Zeppelin. To me there appears to be little of importance in these reflexions, because, as I have tried to argue, the laws of logic are not even a part of the laws of thinking but are the laws of certain objects which can only be grasped by thought.

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Religion and Science: a Philosophical Essay. By JOHN THEODORE MERZ. Edinburgh and London: William Blackwood and Sons, 1915. Pp. xi, 192.

No one has laid the English student of modern thought under heavier or more varied obligations than Dr. Merz, whose sympathetic knowledge is as much in evidence in the present Essay as ever, with the advantage of being set forth in a style of exposition if possible still more lucid and equable than before. He has addressed himself to "the increasing class of thoughtful persons, especially among the younger generation . . . who feel themselves sore perplexed by the contradictions which apparently exist between the dicta of science and the tenets of religious creeds, who are not prepared to sacrifice the truth of either, but who find it extremely difficult to reconcile them" (p. 4).

The brief work is described in the sub-title as "philosophical," but more than once in the text the epithet "psychological" is used rather markedly, as on p. 166, where we read, "the psychological theory developed in the foregoing pages". Not only so, but after an introductory discussion of the ordinary popular view of the outer and the inner world—the view, that is to say, which contrasts these two worlds and puts them in opposition to each other, like the image in a mirror facing its original—Dr. Merz argues that they may better be regarded as "lying, as it were, on the same plane, making up together the total field of our consciousness". Immediately afterwards this is designated the "exclusively introspective point of view"; and the opinion is expressed that the advance of philosophic thought has been retarded by the difficulty of confining oneself strictly to introspective data, though British philosophy more than any other has tended to revert to the true path. And the object of the Essay is stated to be that of applying "this purely introspective view" to a special problem—the problem of Religion.

It is obvious that grave difficulties are involved in this general